

**THE INTRINSIC GEOMETRY OF THE OSCULATING STRUCTURES
THAT UNDERLIE THE HEISENBERG CALCULUS (OR WHY THE
TANGENT SPACE IN SUB-RIEMANNIAN GEOMETRY IS A GROUP)**

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Abstract

We explore the geometry that underlies the osculating structures of the Heisenberg calculus. For a smooth manifold M with a distribution $H \subseteq TM$ analysts have developed explicit (and rather complicated) coordinate formulas to define the nilpotent groups that are central to the calculus. Our aim is, specifically, to gain insight in the *intrinsic* structures that underlie these coordinate formulas. There are two key ideas. First, we construct a certain generalization of the notion of tangent vectors, called “parabolic arrows”, involving a mix of first and second order derivatives. Parabolic arrows are the natural elements for the nilpotent groups of the osculating structure. Secondly, we formulate the natural notion of exponential map for the fiber bundle of parabolic arrows, and show that it explains the coordinate formulas of osculating structures. The result is a conceptual simplification and unification of the treatment of the Heisenberg calculus found in the analytic literature. As a bonus we obtain insight in how to construct a tangent groupoid for this calculus (for arbitrary filtered manifolds), which is a key tool in the study of hypoelliptic Fredholm index problems.

1. INTRODUCTION: OSCULATING NILPOTENT GROUPS IN ANALYSIS AND GEOMETRY

1.1. Motivation. The immediate motivation for this work springs from our investigation of the index problem for Fredholm operators in the Heisenberg calculus. Index theory establishes a bridge between analysis and topology. An initial and difficult obstacle we faced in our investigations was that, in order to derive topological index formulas, we needed a better geometric understanding of the osculating structures that are at the heart of the Heisenberg calculus. We believe that the clarification we achieved in this context is of independent interest, and in the present paper we present our results without reference to index theoretical concerns. This introduction is intended as an exposition of the history of the problem.

1.2. Nilpotent groups in analysis. Osculating structures were first introduced by Gerald Folland and Elias Stein [6] as an aid in the analysis of the tangential CR operator $\bar{\partial}_b$ on the boundary of a strongly pseudoconvex complex domain. Folland and Stein showed how to equip a coordinate chart of a CR manifold M with the structure of a Heisenberg group. Let $H^{1,0} \subset TM \otimes \mathbb{C}$ denote a CR structure on M . For each point $m \in M$, Folland and Stein consider a special type of coordinates $M \supset U \rightarrow \mathbb{R}^{2k+1}$ near m that vary smoothly with m . The coordinate space \mathbb{R}^{2k+1} is identified with the Heisenberg group. The standard left invariant vector fields $X_1, \dots, X_k, Y_1, \dots, Y_k, T$ on the Heisenberg group \mathbb{R}^{2k+1} with their commutation relations

$$[X_i, Y_j] = \delta_{ij}T$$

can be then be identified with vector fields in the chart domain $U \subset M$. Folland and Stein show that the coordinate system can be chosen in such a way that the vector fields

$$Z_j = \frac{1}{2}(X_j + \sqrt{-1}Y_j)$$

are ‘close to’ an orthonormal frame for the bundle $H^{1,0}$. Exactly what it means to be ‘close to’ (or, rather, to ‘osculate’) is made precise in [6, Theorem 14.1]. The point of the construction is that one can approach the hypoellipticity problem for the $\bar{\partial}_b$ operator by studying invariant model operators on the Heisenberg group. The model problem is solved by the techniques of noncommutative harmonic analysis.

Folland and Stein refer to their special coordinate systems as “osculating Heisenberg structures”. Osculating structures of nilpotent groups have played a key role in the subsequent literature on hypoelliptic operators (for example [1, 5, 9, 10]). Following Folland and Stein, analysts typically choose local coordinates $U \rightarrow \mathbb{R}^n$ on an open set $U \subseteq M$, and define a nilpotent group structure on the coordinate space \mathbb{R}^n by means of an explicit formula. For example, in [1] Richard Beals and Peter Greiner introduce osculating structures for *Heisenberg manifolds*.

Heisenberg manifolds are manifolds equipped with a distribution $H \subset TM$ of codimension one. The distribution H may be a contact structure, a foliation, or something more general. Like Folland and Stein, Beals and Greiner start with a system of coordinates that depends on the point $m \in M$, and varies smoothly with m . But now the group structure on the coordinate space \mathbb{R}^{2k+1} is not fixed, but is defined by means of the rather complicated string of formulas (1.8), (1.11), (1.15) in [1]. The formulas for the group structure now involve the derivatives of the smooth coordinate system itself, which plays no role in the definition of Folland and Stein. The nilpotent model groups are no longer necessarily isomorphic to the Heisenberg group, but the calculus is still referred to as “Heisenberg calculus”.

As a final example, in [5] Thomas Cummins generalizes the calculus further to manifolds with a *filtration* $H_1 \subseteq H_2 \subseteq H_3 = TM$, giving rise to three-step nilpotent model groups. His formulas for the osculating structure in this case are very similar to (but more general than) the formulas of Beals and Greiner.

It is hard to discern the intrinsic geometry that underlies these coordinate formulas. The nature of these osculating group structures remains mysterious in the absence of such geometric insight. It is our aim here to provide such insight.

1.3. Nilpotent Lie algebras and the equivalence problem. If one turns to the geometric literature, it turns out that group structures similar to the osculating structures have been around for some time. The oldest mention of such structures is in work on the ‘equivalence problem’ introduced by Cartan in 1910 [3]: Find a full set of infinitesimal invariants of a manifold with distribution $H \subseteq TM$. There exists a simple and elegant definition of the *Lie algebra* of our nilpotent groups. The basic equality,

$$[fX, gY] = fg[X, Y] + f(X.g)Y - g(Y.f)X,$$

shows that if X, Y are sections of H then *modulo* H the value of the bracket $[X, Y](m)$ at $m \in M$ only depends on the values $X(m)$ and $Y(m)$ at m . In other words, the commutator of vector fields induces a *pointwise* bracket,

$$H_m \otimes H_m \rightarrow H_m : X \otimes Y \mapsto [X, Y] \bmod H,$$

where $m \in M$, and $N = TM/H$ denotes the quotient bundle. This can be extended to a Lie bracket on $\mathfrak{g}_m = H_m \oplus N_m$, by taking $[\mathfrak{g}_m, N_m] = 0$. Clearly, the Lie algebra \mathfrak{g}_m is two-step nilpotent. The isomorphism class of these Lie algebras is a first infinitesimal invariant of the distribution H . In this context, the *groups*, if discussed at all, arise only secondarily as the simply-connected exponentials of the graded nilpotent symbol Lie algebras. They don’t play a role in the equivalence problem. (See [7] for a survey of more recent developments in this line of research.)

In any case, this construction provides an intrinsic geometric definition of a bundle of two-step nilpotent Lie algebras for a given distribution $H \subset TM$. And these Lie algebras turn out, of course, to be isomorphic to the Lie algebras of the osculating group structures of the analysts. However, this is not sufficient to clarify the geometry of osculating structures. It is crucial for the analysis of operators on M that we can pull back invariant vector fields (and, more generally, invariant operators) from the osculating groups to the manifold itself. The literature on the equivalence problem contains no hint of how to identify open sets in the manifold with these groups. Clearly, one should take an exponential map $TM \rightarrow M$, and identify $H \oplus N \cong TM$ by choosing a section $N \hookrightarrow TM$. While this is the right general idea, it turns out that the details are rather more subtle.

One of the key ideas in this paper is the identification of the appropriate notion of “exponential map” for the bundle of nilpotent osculating groups. We formulate a geometrically motivated notion of *H-adapted exponential map*, which is at once natural and subtle. The correct notion of *H-adapted exponential map* provides the crucial link between the Lie algebra bundle $H \oplus N$ and the osculating structures of the Heisenberg calculus. But before we can understand the

geometry of H -adapted exponential maps we must clear up one other missing link. What we find in the geometric literature is an intrinsic definition of a bundle of nilpotent *Lie algebras*. But the analysts really need a bundle of nilpotent *groups*. This may not seem like a big deal, especially since the groups are simply connected nilpotent. But the distinction turns out to be crucial.

1.4. Nilpotent groups in sub-Riemannian geometry. It is interesting, in this regard, to consider another geometric context in which this issue arises, namely sub-Riemannian geometry. A sub-Riemannian manifold is a manifold M together with a distribution $H \subset TM$ and a metric on H . In sub-Riemannian geometry the tangent space $T_m M$ at a so-called ‘regular’ point $m \in M$ carries the structure of a nilpotent group. In the special case where $[H, H] = TM$ (brackets of vector fields in H span TM at each point in M) this group is, again, isomorphic to our osculating group. As before, the group structure is defined by exponentiating the Lie algebra structure of $H_m \oplus N_m$, exactly as it is done in work on the equivalence problem. But in sub-Riemannian geometry it is really the *group* structure on $T_m M$ that is of interest, not the Lie algebra structure. It is the tangent space as a *group* that makes it a useful approximation to the manifold *as a sub-Riemannian metric space*. This situation is more closely related to the osculating structures as they appear in analysis.

In [2, p.73–76] André Bellaïche explicitly considers the question, “Why is the tangent space a group?” Bellaïche expresses to have been puzzled by this situation, and that drawing a Lie algebra structure from the bracket structure of vector fields does not seem to be the appropriate answer. What is missing, of course, is a *direct* definition of the group structure on $T_m M$ (or on $H_m \oplus N_m$) that makes it intuitively clear how composition of group elements arises. As we will show in this paper, it is precisely the formulation of a satisfactory answer to this question that will lead us to the appropriate notion of H -adapted exponential maps, which, in turn, fully clarifies the intrinsic geometry underlying the osculating structures of the Heisenberg calculus.

In considering this question, Bellaïche mentions Alain Connes’ *tangent groupoid* as containing a hint of what an answer might look like. The tangent groupoid is obtained by taking the trivial groupoid $M \times M$ and “blowing up the diagonal”. Composition of pairs in $M \times M$ is

$$(a, b) \cdot (c, d) \begin{cases} = (a, d) & \text{if } b = c, \\ \text{not defined} & \text{if } b \neq c. \end{cases}$$

By introducing a parameter $t \in [0, 1]$, we can let the pair $(a(t), b(t))$ converge to a tangent vector. Provided that $a(0) = b(0) = m$ we have

$$v = \lim_{t \rightarrow 0} \frac{a(t) - b(t)}{t} \in T_m M.$$

This defines a topology on the groupoid that is the union

$$TM \cup M \times M \times (0, 1].$$

This is Connes’ tangent groupoid in a nutshell. As Connes shows, the tangent groupoid can be equipped with a natural smooth structure. (See [4], II.5).

This construction is of interest to Bellaïche because it gives a nice way to obtain the tangent space TM , considered as a bundle of abelian groups, as the limit $t \rightarrow 0$ of the trivial groupoid $M \times M$. It thus “explains” how the usual abelian group structure in the tangent fibers arises as the infinitesimal limit of the composition of pairs in $M \times M$. But Bellaïche concludes that “one should not be abused” to “think that the algebraic structure of $T_p M$ stems from the absolutely trivial structure of $M \times M$!” In other words, he does not see how the idea of a tangent groupoid could explain the nilpotent group structure in the tangent fibers. Rather, Bellaïche maintains, the group structure “is concealed in dilations”. To explain this assertion Bellaïche then relates a satisfactory geometric answer to his question communicated to him by Mikhael Gromov. We will not relate the details of Gromov’s solution here. The interested reader is referred to [2].

Instead, we provide in this paper our own answer to Belaïche’s question. As we will show, it is certainly true that the algebraic structure is concealed in dilations, as Gromov explained to Belaïche, but it is *also* correct to say that it stems from the trivial structure of $M \times M$! When we discuss the tangent groupoid that is appropriate for this situation (in section 4), we will argue that the real secret that makes it work (and which makes it hard to find the correct definition) is not the trivial structure of $M \times M$ nor even the use of dilations, but rather the appropriate notion of H -adapted exponential map. In other words, one must identify exactly what it means that the groups ‘osculate’ the manifold.

1.5. Overview of the paper. In section 2 we define a new kind of “tangent vector”—or, rather, a generalization of tangent vectors that is appropriate for an understanding of osculating structures. We call these new objects “parabolic arrows”. (We would have liked to call them “parabolic vectors”, but that term would falsely suggest that they are a special type of vector.) Like a tangent vector, a parabolic arrow is an infinitesimal approximation of a smooth curve near a point. But it is an infinitesimal approximation that is suited to the “parabolic dilations” that are key to understanding the Heisenberg calculus. While tangent vectors are defined by means of first order derivatives, parabolic arrows involve a mix of first and second order derivatives.

As we will see in section 2, parabolic arrows can be composed in a natural way by extending them to local *flows*, which compose in the obvious way. This is a natural generalization of how addition of tangent vectors could be defined, but the appearance of second order derivatives significantly complicates the picture. In particular, composition of parabolic arrows is noncommutative. Parabolic arrows at a point $m \in M$ are shown to form a nilpotent Lie group, and the Lie algebra $H_m \oplus N_m$ defined by taking brackets of vector fields is shown to be its Lie algebra. This shows that parabolic arrows are indeed a geometric realization of elements in the osculating group.

Armed with new geometric insight in the nature of the group elements and their composition, we introduce in section 3 the appropriate notion of an exponential map for the fiber bundle of parabolic arrows. Because parabolic arrows are related to curves on the manifold, the group of parabolic arrows has a built-in connection to the manifold. Because of this connection a definition of exponential map for parabolic arrows suggests itself naturally. We analyze the osculating structures of Folland and Stein for CR manifolds and of Beals and Greiner for Heisenberg manifolds, and we *derive* the explicit coordinate formulas used by these analysts from our intrinsic geometric concepts.

Finally, in section 4—counter to Belaïche’s suggestion to the contrary—we will see that our construction of the osculating groups blends perfectly with the tangent groupoid formalism. We employ parabolic arrows and H -adapted exponential maps to construct a tangent groupoid for the Heisenberg calculus. We constructed such a groupoid for the special case of contact manifolds in [11], making use of Darboux’s theorem. In section 4 we will show how parabolic arrows and H -adapted exponential maps make the construction of a tangent groupoid for the Heisenberg calculus a straightforward generalization of Connes’ construction. (Our construction here applies in the case of an *arbitrary* distribution $H \subseteq TM$. In the special case of Heisenberg manifolds an alternative construction of this groupoid was given by Raphael Ponge in [8].)

1.6. Summary picture of the Heisenberg calculus. To conclude this introduction, let us summarize the simplified picture of the Heisenberg calculus as it emerges from our consideration here.

Starting with a distribution $H \subseteq TM$ we have a filtration of the Lie algebra $\Gamma(TM)$ of vector fields on M , where sections in H are given order 1 and all other sections in TM have order 2. The associated *graded* Lie algebra can be identified with sections in the bundle of two-step graded Lie algebras $H \oplus N$, whose construction was explained above. Let $T_H M$ denote the bundle of nilpotent Lie *groups* associated to $H \oplus N$, which we may identify with the fiber bundle of parabolic arrows.

Choose a section $j: N \hookrightarrow TM$ and the corresponding identification $j: H \oplus N \rightarrow TM$. We may then identify $T_H M$ with TM as smooth fiber bundles. Also choose a connection ∇ on TM . The crux of the matter is that the connection ∇ must be chosen so as to *preserve the distribution* H . In other words, ∇ must be a connection compatible with the G -structure on TM that corresponds to the choice of a distribution $H \subseteq TM$. As we will see such an exponential map $\exp^\nabla: TM \rightarrow M$ induces an H -adapted exponential map

$$\exp^\nabla: T_H M \rightarrow M.$$

Extend this map to a local diffeomorphism near the zero section in $T_H M$,

$$h: T_H M \rightarrow M \times M: (m, v) \mapsto (\exp_m^\nabla(\delta_t v), m),$$

where δ_t are the natural dilations of the graded nilpotent osculating groups. By means of the map h we can then pull back the Schwartz kernel $k(m, m')$ of a linear operator $C^\infty(M) \rightarrow \mathcal{D}(M)$ from $M \times M$ to the bundle of osculating groups $T_H M$. The pull back $h^*k = k \circ h$ constitutes a smooth family $k_m(v) = k(h(m, v))$ of distributions on the nilpotent groups $T_H M_m$ parametrized by $m \in M$. If the operator associated to k is pseudolocal, then each distribution k_m has an isolated singularity at the origin of the group $T_H M_m$.

Notice the complete absence, in this account, of anything resembling the coordinate formulas (1.8), (1.11), (1.15) in [1]. Also observe that the description is valid for *arbitrary* distributions $H \subseteq TM$.

From here on the calculus can be developed as usual. An operator is in the calculus if the distributions h^*k has an asymptotic expansion for the dilations on $T_H M$, etcetera. It follows from the smoothness of the parabolic tangent groupoid that the highest order part in this expansion is invariantly defined, independent of the choice of H -adapted exponential map. To see this, one merely needs to recognize that the highest order part in the expansion of h^*k on $T_H M$ is precisely the limit, as $t \downarrow 0$, of the distribution $t^d k$ on $M \times M \times (0, 1]$, which is obviously invariantly defined. (Here d is the order of the operator.)

Remark. We have focused in this paper on the Heisenberg calculus for manifolds equipped with a single distribution $H \subseteq TM$, which is modeled by analysis on *two-step* nilpotent groups. But the Heisenberg calculus has been generalized to arbitrary *filtered manifolds*. A filtered manifold is a manifold with a nested sequence of distributions

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_r = TM,$$

where it is required that the sections in these bundles form a filtration on the Lie algebra of vector fields, i.e.,

$$[\Gamma(H_i), \Gamma(H_j)] \subseteq \Gamma(H_{i+j}).$$

In [5] Cummins develops the details of such a calculus for the case $r = 3$, modeled on three-step nilpotent groups. The literature also contains references to a preprint by Anders Melin entitled *Filtered Lie-algebras and pseudodifferential operators* (see, for example, the references in [10]). This paper reportedly developed the calculus in its most general form for arbitrary filtered manifolds, modeled on graded nilpotent groups of arbitrary length. The preprint is from 1982, but it seems that it has never been published. Nevertheless, without having access to this preprint it is perfectly clear from the ideas developed here how one could procede. Everything that has been said above about the two-step situation generalizes easily to this most general case. We do not know the details of Melins calculus, but in our approach one simply chooses a connection ∇ on TM that preserves each of the distributions H_j . Everything else is then basically the same as before.

2. PARABOLIC ARROWS AND THEIR COMPOSITION.

2.1. Parabolic arrows. Throughout this and the following sections, M denotes a smooth manifold with a specified distribution $H \subseteq TM$. We will write $N = TM/H$ for the quotient bundle, and denote the fiber dimensions by $p = \dim H$, $q = \dim N$, and $n = p + q = \dim M$. We will *not* assume that $q = 1$.

When studying a Heisenberg structure (M, H) it is convenient to work with a special type of coordinates.

Definition 1. Let m be a point on M , and $U \subseteq M$ an open set in M containing m . A coordinate chart $\phi: U \rightarrow \mathbb{R}^n$, $\phi(m') = (x_1, \dots, x_n)$ is called an H -coordinate chart at m , if $\phi(m) = 0$, and the first p coordinate vectors $\partial/\partial x_i$ ($i = 1, \dots, p$) at the point m span the fiber H_m of H at m .

Tangent vectors can be defined as equivalence classes of smooth curves. By analogy, we introduce an equivalence relation involving second-order derivatives.

Definition 2. Let $c_1, c_2: [-1, 1] \rightarrow M$ be two smooth curves that are tangent to H at $t = 0$. For such curves we say that $c_1 \sim_H c_2$ if $c_1(0) = c_2(0)$ and if, choosing H -coordinates centered at $c_1(0) = c_2(0)$, we have

$$\begin{aligned} c'_1(0) - c'_2(0) &= 0, \\ c''_1(0) - c''_2(0) &\in H. \end{aligned}$$

An equivalence class $[c]_H$ is called a parabolic arrow at the point $c(0)$. The set of parabolic arrows at $m \in M$ is denoted $T_H M_m$, while

$$T_H M = \bigcup_{m \in M} T_H M_m.$$

We can give $T_H M$ the topology induced by the C^2 -topology on the set of curves, but for the moment we just think of $T_H M$ as a set.

Lemma 3. The equivalence relation \sim_H is well-defined, i.e., independent of the choice of the H -coordinates.

Proof. The condition that $c'_1(0) = c'_2(0)$ is clearly invariant. We will show that, assuming $c'_1(0) = c'_2(0)$, the condition $c''_1(0) - c''_2(0) \in H$ on the second derivatives is independent of the choice of H -coordinates.

If ψ is a change of H -coordinates, then:

$$\begin{aligned} \frac{d^2(\psi \circ c)}{dt^2} &= \frac{d}{dt} \left(\sum_j \frac{\partial \psi}{\partial x_j}(c(t)) \frac{dc^j}{dt} \right) \\ &= \sum_{j,k} \frac{\partial^2 \psi}{\partial x_j \partial x_k}(c(t)) \frac{dc^j}{dt} \frac{dc^k}{dt} + \sum_j \frac{\partial \psi}{\partial x_j}(c(t)) \frac{d^2 c^j}{dt^2}. \end{aligned}$$

At $t = 0$ we assumed $dc_1/dt = dc_2/dt$, so that the first term on the right hand side is equal for $\psi \circ c_1$ and $\psi \circ c_2$ (at $t = 0$). Therefore:

$$(\psi \circ c_1)''(0) - (\psi \circ c_2)''(0) = \frac{\partial \psi}{\partial x}(m) \cdot (c''_1(0) - c''_2(0)).$$

Since ψ is a change of H -coordinates at m , $\partial \psi / \partial x$ preserves H_m , so that $c''_1(0) - c''_2(0) \in H_m$ implies $(\psi \circ c_1)''(0) - (\psi \circ c_2)''(0) \in H_m$. □

If we fix H -coordinates at $m \in M$, and consider the second-order expansion (in coordinates) of a curve c with $c(0) = m$,

$$c(t) = c'(0)t + \frac{1}{2}c''(0)t^2 + \mathcal{O}(t^3),$$

we see that any such curve is equivalent, as a parabolic arrow, to a curve \tilde{c} of the form

$$\tilde{c}(t) = (th, t^2n) = (th_1, \dots, th_p, t^2n_1, \dots, t^2n_q).$$

This observation forms the basis for the following definition.

Definition 4. Suppose we are given H -coordinates at $m \in M$. Let $h \in \mathbb{R}^p, n \in \mathbb{R}^q$, and let $c(t)$ be the curve in M defined (in H -coordinates) by

$$c(t) = (th, t^2n).$$

We call $(h, n) = (h_1, \dots, h_p, n_1, \dots, n_q) \in \mathbb{R}^{p+q}$ the Taylor coordinates for the parabolic arrow $[c]_H \in T_H M_m$, induced by the given H -coordinates at m ,

$$F_m : \mathbb{R}^{p+q} \rightarrow T_H M_m : (h, n) \mapsto [c]_H.$$

This is analogous to the way in which coordinates on the tangent space $T_m M$ are induced by coordinates on M , with the important difference that Taylor coordinates on $T_H M_m$ are defined for only one fiber (i.e., one point $m \in M$) at a time.

Analogous to the directed line segments that represent tangent vectors, a pictorial representation for the class $[c]_H$ would be a directed segment of a parabola. Hence our name ‘parabolic arrow.’ Parabolic arrows are what smooth curves look like infinitesimally, when we blow up the manifold using the dilations $(h, n) \mapsto (th, t^2n)$, and let $t \rightarrow \infty$.

When working with H -coordinates $\phi(m) = x \in \mathbb{R}^n$, we use the notation $x = (x^H, x^N) \in \mathbb{R}^{p+q}$, where

$$x^H = (x_1, \dots, x_p) \in \mathbb{R}^p, x^N = (x_{p+1}, \dots, x_{p+q}) \in \mathbb{R}^q.$$

Lemma 5. If ψ is a change of H -coordinates at m , then the induced change of Taylor coordinates $\psi(h, n) = (h', n')$ for a given parabolic vector in $T_H M_m$ is given by the quadratic formula:

$$\begin{aligned} h' &= D\psi(h), \\ n' &= [D\psi(n) + D^2\psi(h, h)]^N, \end{aligned}$$

where $[v]^N$ denotes the normal component of the vector $v = (v^H, v^N) \in \mathbb{R}^{p+q}$.

Proof. This is just the formula for $(\psi \circ c)''(0)$ from the proof of Lemma 3. □

Corollary 6. The smooth structures on the set of parabolic arrows $T_H M_m$ at a point $m \in M$ defined by different Taylor coordinates are compatible, i.e., $T_H M_m$ has a natural structure of a smooth manifold.

It is clear from Lemma 5 that Taylor coordinates define a structure on $T_H M_m$ that is more than just a smooth structure. This will be fully clarified when we introduce the *group* structure on $T_H M_m$, but part of this extra structure is captured if we consider how parabolic arrows behave when rescaled.

Definition 7. The family of dilations $\delta_s, s > 0$, on the space of parabolic arrows $T_H M_m$ is defined by

$$\delta_s([c]_H) = [c_s]_H,$$

where $[c]_H$ is a parabolic arrow in $T_H M_m$ represented by the curve $c(t)$, and c_s denotes the reparametrized curve $c_s(t) = c(st)$.

When working in Taylor coordinates $[c]_H = (h, n)$, we simply have

$$\delta_s(h, n) = (sh, s^2n).$$

Clearly, these dilations are smooth maps and $\delta_{st} = \delta_s \circ \delta_t$.

Considering Taylor coordinates on $T_H M_m$, it is tempting to identify parabolic arrows with vectors in $H \oplus N$. Lemma 5 shows that such an identification is not invariant if we use Taylor coordinates to define it. But we have at least the following result.

Lemma 8. *There is a natural identification*

$$T_0(T_H M_m) \cong H_m \oplus N_m$$

of the tangent space $T_0(T_H M_m)$ at the ‘origin’ (i.e., at the equivalence class $[0]_H$ of the constant curve at m) with the vector space $H_m \oplus N_m$. It is obtained by identifying the coordinates on $T_0(T_H M_m)$ induced by Taylor coordinates on $T_H M_m$, with the natural coordinates on $H_m \oplus N_m$.

Proof. From Lemma 5, we see that Taylor coordinates on $T_0(T_H M_m)$ transform according to the formula

$$h' = D\psi(h), \quad n' = D\psi(n)^N,$$

because the quadratic term $D^2\psi(h, h)$ has derivative 0 at 0. This is precisely how the induced coordinates on $H_m \oplus N_m$ behave under coordinate transformation ψ . □

2.2. Composition of parabolic arrows. We will now show that the manifold $T_H M_m$ has the structure of a Lie group. Our method is based on composition of local flows of M . By a flow Φ of M we mean a diffeomorphism $\Phi: M \times \mathbb{R} \rightarrow M$, such that $m \mapsto \Phi_t(m) = \Phi(m, t)$ is a diffeomorphism for each $t \in \mathbb{R}$, while $\Phi_0(m) = m$. A *local flow* is only defined on an open subset $V \subseteq M \times \mathbb{R}$. Two flows Φ, Ψ can be composed:

$$(\Phi \circ \Psi)(m, t) = \Phi(\Psi(m, t), t).$$

Using the notation Φ_t for the local diffeomorphism $\Phi_t(m) = \Phi(m, t)$, we can write $(\Phi \circ \Psi)_t = \Phi_t \circ \Psi_t$.

A (local) flow is said to be generated by the vector field $X \in \Gamma(TM)$, if

$$\frac{\partial \Phi}{\partial t}(m, t) = X(m).$$

However, we are specifically interested in flows for which the generating vector field $X_t(m) = \frac{\partial \Phi}{\partial t}(m, t)$ is *not constant*, but depends on t . We will only require that X_0 is a section in H , but we will allow X_t to pick up a component in the N -direction. This is because we are not primarily interested in the *tangent vectors* to the flow lines $c_m(t) = \Phi(m, t)$, but in the *parabolic arrows* that they define.

We start with a formula that gives a quadratic approximation (in t) for the composition of two arbitrary flows.

Lemma 9. *Let Φ^X, Φ^Y be two flows in \mathbb{R}^n that are defined near the origin, and let X and Y be their generating vector fields at $t = 0$:*

$$X(x) = (\partial_t \Phi^X)(x, 0), \text{ and } Y(x) = (\partial_t \Phi^Y)(x, 0).$$

Then the composition of Φ^X and Φ^Y has the following second-order approximation,

$$(\Phi_t^X \circ \Phi_t^Y)(0) = \Phi_t^X(0) + \Phi_t^Y(0) + t^2 (\nabla_Y X)(0) + \mathcal{O}(t^3),$$

where ∇ denotes the standard connection on $T\mathbb{R}^n$.

Remark. Observe that $X = \partial_t \Phi^X$ is required only at $t = 0$!

Proof. Write $F(r, s) = \Phi_r^X(\Phi_s^Y(0))$. The Taylor series for F gives

$$\begin{aligned} F(t, t) &= t \partial_r F(0, 0) + t^2 \partial_s \partial_r F(0, 0) + t \partial_s F(0, 0) + \frac{1}{2} t^2 \partial_r^2 F(0, 0) \\ &\quad + \frac{1}{2} t^2 \partial_s^2 F(0, 0) + \mathcal{O}(t^3) \\ &= F(t, 0) + F(0, t) + t^2 \partial_s \partial_r F(0, 0) + \mathcal{O}(t^3), \end{aligned}$$

or

$$\Phi_t^X \Phi_t^Y(0) = \Phi_t^X(0) + \Phi_t^Y(0) + t^2 \partial_s \partial_r \Phi_r^X \Phi_s^Y(0)|_{r=s=0} + \mathcal{O}(t^3).$$

At $r=0$ we have $\partial_r \Phi_r^X = X$, so:

$$\partial_s \partial_r \Phi_r^X(\Phi_s^Y(0))|_{r=0} = \partial_s (X(\Phi_s^Y(0))).$$

Here $X(\Phi_s^Y(0))$ denotes the vector field X evaluated at the point $\Phi_s^Y(0)$, which can be thought of as a point on the curve $s \mapsto \Phi_s^Y(0)$. The operator ∂_s is applied to the components of this vector, and the chain rule gives

$$\partial_s X(\Phi_s^Y(0))|_{s=0} = \sum_i \partial_i X(0) \cdot \partial_s \Phi_s^Y(0)^i|_{s=0} = \sum_{i=1}^p (\partial_i X)(0) Y^i(0) = (\nabla_Y X)(0).$$

□

We are interested in flows Φ for which the *flow lines* $\Phi^m(t) = \Phi(m, t)$ define parabolic arrows. Hence the following definition.

Definition 10. A parabolic flow of (M, H) is a local flow $\Phi: V \rightarrow M$ (with V an open subset in $M \times \mathbb{R}$) whose generating vector field at $t = 0$,

$$\frac{\partial \Phi}{\partial t}(m, 0)$$

(defined at each point m for which $(m, 0) \in V$) is a section of H .

Given a parabolic flow Φ , each of the flow lines Φ^m is tangent to H at $t = 0$, and so determines a parabolic arrow $[\Phi^m]_H$ at each $m \in M$ (with $(m, 0) \in V$). Once we have defined the smooth structure on $T_H M$ it will become clear that $m \mapsto \Phi^m$ is a smooth section of the bundle $T_H M$. It is an analogue of the notion of a generating vector field, but it is only defined at $t = 0$.

We now show how composition of parabolic flows induces a group structure on the fibers of $T_H M$.

Proposition 11. Let Φ, Ψ be two parabolic flows. Then the composition $(\Phi \circ \Psi)_t = \Phi_t \circ \Psi_t$ (defined on an appropriate domain) is also a parabolic flow, and the parabolic vector $[(\Phi \circ \Psi)^m]_H$ at a point $m \in M$ only depends on the parabolic vectors $[\Phi^m]_H$ and $[\Psi^m]_H$ at the same point.

Proof. Let ∇ denote the standard local connection on TM induced by the H -coordinates at m . Because $\nabla_{fY}(gX) = f \nabla_Y(gX) + g \nabla_Y(fX)$, we see that the operation

$$\Gamma^\infty(H) \otimes \Gamma^\infty(H) \rightarrow \Gamma^\infty(N) : (X, Y) \mapsto [\nabla_Y X]^N$$

is $C^\infty(M)$ -bilinear. In other words, the N -component of $\nabla_Y X$ at the point $m \in M$ only depends on the values $X(m)$ and $Y(m)$ at m . We denote this N -component by ∇^N :

$$\begin{aligned} \nabla^N : H_m \otimes H_m &\rightarrow N_m, \\ \nabla^N(X(m), Y(m)) &= [\nabla_Y X]^N(m). \end{aligned}$$

Lemma 9 implies:

$$\begin{aligned}\Phi_t \Psi_t(0)^H &= \Phi_t(0)^H + \Psi_t(0)^H + \mathcal{O}(t^2), \\ \Phi_t \Psi_t(0)^N &= \Phi_t(0)^N + \Psi_t(0)^N + t^2 \nabla^N(X(0), Y(0)) + \mathcal{O}(t^3).\end{aligned}$$

Writing

$$\begin{aligned}\Phi_t(0)^H &= th + \mathcal{O}(t^2), \quad \Phi_t(0)^N = t^2 n + \mathcal{O}(t^3), \\ \Psi_t(0)^H &= th' + \mathcal{O}(t^2), \quad \Psi_t(0)^N = t^2 n' + \mathcal{O}(t^3),\end{aligned}$$

this becomes

$$\begin{aligned}\Phi_t \Psi_t(0)^H &= t(h + h') + \mathcal{O}(t^2), \\ \Phi_t \Psi_t(0)^N &= t^2 (n + n' + \nabla^N(h, h')) + \mathcal{O}(t^3).\end{aligned}$$

The proposition is a direct corollary of these formulas. \square

It is clear from Proposition 11 that composition of parabolic flows induces a group structure on the set $T_H M_m$, for each $m \in M$, analogous to addition of tangent vectors in $T_m M$. To see that $T_H M_m$ is actually a *Lie group*, we use the explicit formulas obtained in the proof of Proposition 11.

Proposition 12. *Let Φ, Ψ be two parabolic flows. Given H -coordinates at m , let $X_i \in \Gamma(H)$ ($i = 1, \dots, p$) be local sections in H that extend the coordinate tangent vectors ∂_i at m . Let X_i^l ($l = 1, \dots, n$) denote the coefficients of the vector field X_i , i.e.,*

$$X_i = \sum X_i^l \partial_l.$$

Let (b_{ij}^k) be the array of constants

$$b_{ij}^k = \partial_j X_i^{p+k}(m)$$

for $i, j = 1, \dots, p$ and $k = 1, \dots, q$. It represents a bilinear map $b: \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ via

$$b(v, w)^k = \sum_{i,j=1}^p b_{ij}^k v^i w^j,$$

with $k = 1, \dots, q$.

If (h, n) and (h', n') are the Taylor coordinates of $[\Phi^m]_q$ and $[\Psi^m]_q$, respectively, then the Taylor coordinates (h'', n'') of $[(\Phi \circ \Psi)_m]_q$ are given by

$$\begin{aligned}h'' &= h + h', \\ n'' &= n + n' + b(h, h').\end{aligned}$$

Proof. This is a direct corollary of the formulas in the proof of Proposition 11. Simply observe that

$$b(\partial_i, \partial_j)^k = \partial_j X_i^{p+k}(m) = [\nabla_{X_j} X_i]^{p+k}(m) = \nabla(\partial_i, \partial_j)^{p+k},$$

which implies that $b(v, w) = \nabla^N(v, w)$. \square

Corollary 13. *The operation*

$$[\Phi^m]_H * [\Psi^m]_H = [(\Phi \circ \Psi)^m]_H$$

defines the structure of a Lie group on $T_H M_m$.

The groups $T_H M_m$ are the *osculating groups* associated to the distribution (M, H) . Note that, although the value of the array (b_{ij}^k) in Proposition 12 depends on the choice of coordinates, our definition of the group elements (parabolic vectors) as well as their composition is clearly coordinate-independent. Furthermore, it is important to notice that the values of b_{ij}^k do *not* depend on the choice of vector fields X_i , but only on the choice of coordinates.

Beals and Greiner *defined* the osculating groups by means of the formulas we have derived in Proposition 12 (see [1], chapter 1). Note that in their treatment $q = 1$, so that the k -index in the array b_{ij}^k is missing. The osculating group itself was simply identified with the coordinate space \mathbb{R}^n , and its status as an independent geometric object was left obscure.

Proposition 14. *The natural dilations δ_s of the osculating groups $T_H M_m$, induced by reparametrization of curves, are Lie group automorphisms.*

Proof. That the dilations are group automorphisms follows immediately from the geometric definition of the group operation on $T_H M_m$ in Corollary 13 (by reparametrizing the flows). Alternatively, using Taylor coordinates we have $\delta_s(h, n) = (sh, s^2n)$, which is clearly a smooth automorphism for the group operation

$$(h, n) * (h', n') = (h + h', n + n' + b(h, h')).$$

□

The construction of the osculating bundle $T_H M$ is *functorial* for (local) diffeomorphisms. Given a diffeomorphism of manifolds with distribution,

$$\phi : (M, H) \rightarrow (M', H'),$$

such that $D\phi : H \rightarrow H'$, we could define the *parabolic derivative* $T_H \phi$ of ϕ as the map

$$T_H \phi : T_H M \rightarrow T_H M' : [c]_H \mapsto [\phi \circ c]_{H'},$$

where $c(t)$ is a curve in M representing a parabolic arrow $[c]_H \in T_H M_m$. A straightforward calculation, similar to the proof of Lemma 3, shows that this is a well-defined map (independent of the choice of the curve c), and functoriality is obvious, i.e.,

$$T_H(\phi \circ \phi') = T_H \phi \circ T_H \phi'.$$

Clearly, if ϕ is a diffeomorphism, then $T_H \phi$ is a group isomorphism in each fiber.

2.3. The Lie algebras of the osculating groups. According to Lemma 8, we may identify the Lie algebra $\text{Lie}(T_H M_m)$ as a vector space with $H_m \oplus N_m$. In the introduction we defined a Lie algebra structure on $H_m \oplus N_m$, and we now show that it is compatible with the group structure on $T_H M_m$. We make use of some general results on two-step nilpotent groups that are discussed in the appendix.

Proposition 15. *Let X and Y be two (local) sections of H . Then the value of the normal component $[X, Y]^N(m)$ of the bracket $[X, Y]$ at the point m only depends on the values of X and Y at the point m .*

The Lie algebra structure on $\text{Lie } T_H M_m \cong H_m \oplus N_m$ is given by

$$[(h, n), (h', n')] = (0, [X, Y]^N(m)),$$

where $X, Y \in \Gamma(H)$ are arbitrary vector fields with $X(m) = h, Y(m) = h'$.

Proof. This is a straightforward application of Lemma 25 to the group structure on $T_H M_m$ as described in Proposition 11. We have

$$b(h', h) - b(h, h') = (\nabla_X Y - \nabla_Y X)^N(m) = [X, Y]^N(m).$$

We have already shown that $b(h, h') = (\nabla_Y X)^N(m)$ only depends on $h = X(m)$ and $h' = Y(m)$. □

We are now in a position to define the smooth structure on the total space

$$T_H M = \bigcup T_H M_m.$$

There is a natural bijection

$$\exp : H_m \oplus N_m \rightarrow T_H M_m,$$

namely the exponential map from the Lie algebra $H_m \oplus N_m$ to the Lie group $T_H M_m$. We give the total space $T_H M$ the smooth structure that it derives from its identification with $H \oplus N$.

Lemma 16. *The smooth structure on $T_H M$, obtained by the fiberwise identification with $H \oplus N$ via exponential maps, is compatible with the Taylor coordinates on each $T_H M_m$, for any choice of H -coordinates at m .*

Proof. Choosing H -coordinates at m , we get linear coordinates on $H_m \oplus N_m$. Taking these coordinates and Taylor coordinates on $T_H M_m$, we have identified $\text{Lie } T_H M_m \cong H_m \oplus N_m$. According to Proposition 24, the exponential map $H_m \oplus N_m \rightarrow T_H M_m$ is expressed in these coordinates as

$$\exp(h, n) = (h, n + \frac{1}{2}b(h, h)),$$

which is clearly a diffeomorphism. □

The natural decomposition $\text{Lie}(T_H M_m) = H_m \oplus N_m$ defines a Lie algebra grading, with $\mathfrak{g}_1 = H_m$ of degree 1 and $\mathfrak{g}_2 = N_m$ of degree 2. Corresponding to the grading we have dilations $\delta_t(h, n) = (th, t^2n)$, and these dilations are Lie algebra automorphisms. The dilations of the osculating group $T_H M_m$ induced by reparametrization of curves and the dilations of the graded Lie algebra $H_m \oplus N_m$ are related via the exponential map (see Proposition 24):

$$\exp(\delta_t(h, n)) = \exp(th, t^2n) = (th, t^2n + \frac{1}{2}b(th, th)) = \delta_t(h, n + \frac{1}{2}b(h, h)) = \delta_t \exp(h, n).$$

It will be useful to characterize the parabolic arrows whose logarithms are vectors in H .

Proposition 17. *If $c: \mathbb{R} \rightarrow M$ is a curve such that $c'(t) \in H$ for all $t \in (-\varepsilon, \varepsilon)$, then the parabolic arrow $[c]_H \in T_H M_m$ is the exponential of the tangent vector $c'(0) \in H_m$, considered as an element in the Lie algebra of $T_H M_m$. Here $m = c(0)$.*

Proof. Choose H -coordinates at $m = c(0)$, and let $(h, n) \in \mathbb{R}^{p+q}$ be the corresponding Taylor coordinates of the parabolic arrow $[c]_H$. Because $c'(t) \in H$ for t near 0, we can choose an H -frame X_1, \dots, X_n in a neighborhood U of m in such a way that $c'(t) = \sum h_i X_i(c(t))$ at every point $c(t) \in U$. With this set up, we compute the second derivative:

$$\frac{d^2 c}{dt^2} = \frac{d}{dt} \left(\sum_i h_i X_i \right) \circ c = \sum_j \frac{\partial}{\partial x_j} \left(\sum_i h_i X_i \right) \frac{dc_j}{dt} = \sum_{i,j} h_i \frac{\partial X_j}{\partial x_i} \frac{dc_j}{dt}.$$

Then, at $t = 0$, the normal component of $c''(0)$ is given by,

$$c''(0)^N = \sum_{i,j} \partial_i X_j^N h_i h_j = b(h, h),$$

where $b(h, h)$ is defined as in Proposition 12. It follows that the Taylor coordinates of $[c]_H$ are $(h, \frac{1}{2}b(h, h))$, and therefore, by Proposition 24,

$$\log([c]_H) = (h, 0).$$

□

3. H -ADAPTED EXPONENTIAL MAPS.

3.1. Exponential maps for parabolic arrows. We now come to the definition of *exponential map* that is suitable for the fiber bundle of parabolic arrows. We will see that such an exponential map is equivalent to what analysts mean by an osculating structure. Our Definition 19 of H -adapted exponential maps clarifies the intrinsic geometry behind the explicit coordinate formulas for osculating structures proposed by analysts.

Before introducing our notion of H -adapted exponential map, let us first state exactly what we mean by ‘exponential map’.

Definition 18. *An exponential map is a smooth map*

$$\exp: TM \rightarrow M$$

whose restriction $\exp_m: T_m M \rightarrow M$ to a fiber $T_m M$ maps $0 \mapsto m$, while the derivative $D\exp_m$ at the origin $0 \in T_m M$ is the identity map $T_0(T_m M) = T_m M \rightarrow T_m M$.

A more specialized notion of ‘exponential map’ associates it to a connection ∇ on TM . For a vector $v \in T_m M$ one can define $\exp_m^\nabla(v)$ to be the end point $c(1)$ of the unique curve $c(t)$ that satisfies

$$c(0) = m, \quad c'(0) = v, \quad \nabla_{c'(t)} c'(t) = 0.$$

Such a map certainly satisfies the property of Definition 18. One could specialize further and let ∇ be the Levi-Civita connection for a Riemannian metric on M , in which case the curve c will be a geodesic. But the looser Definition 18 suffices for the purpose of defining a pseudodifferential calculus on M . Given such an exponential map, we can form a diffeomorphism h in a neighborhood of the zero section $M \subset TM$,

$$h: TM \rightarrow M \times M: (m, v) \mapsto (\exp_m(v), m).$$

Given the Schwartz kernel $k(m, m')$ of a linear operator $C^\infty(M) \rightarrow \mathcal{D}(M)$, we can pull it back to a distribution on TM by means of the map h . The singularities of the pullback $k \circ h$ reside on the zero section $M \in TM$. The classical pseudodifferential calculus is obtained by specifying the asymptotic expansion of $k \circ h$ near the zero section. In order to develop the Heisenberg pseudodifferential calculus in an analogous manner we must identify the natural notion of exponential map for the fiber bundle $T_H M$ of parabolic arrows. Our geometric insight in the group structure of $T_H M$ leads to a natural definition. Observe that Definition 18 is equivalent to the condition that each curve $c(t) = \exp_m(tv)$ has tangent vector $c'(0) = v$ at point $m = c(0)$. This immediately suggests the following generalization.

Definition 19. *Let M be a manifold with distribution $H \subset TM$. An H -adapted exponential map for (M, H) is a smooth map*

$$\exp: T_H M \rightarrow M,$$

such that for each parabolic arrow $v \in T_H M_m$ the curve $c(t) = \exp_m(\delta_t v)$ in M represents the parabolic arrow v , i.e., $[c]_H = v$ in $T_H M_m$.

If we choose a section $j: N \hookrightarrow TM$ we may identify $H \oplus N$ with the tangent bundle TM . The composition

$$T_H M \xrightarrow{\log} H \oplus N \xrightarrow{j} TM,$$

then identifies $T_H M$, as a smooth fiber bundle, with TM . Every H -adapted exponential map is identified, in this way, with an ordinary exponential map $\exp: TM \rightarrow M$. However, not every exponential map $TM \rightarrow M$ induces an H -adapted exponential map. Definition 18 involves only the *first derivative* of the map. But in order to be H -adapted, an exponential map must satisfy

a further requirement on its *second derivative*. Definition 19 provides a natural geometric way to encode this rather delicate second order condition, by means of our notion of parabolic arrows.

The key property that makes an arbitrary exponential map H -adapted can be stripped down to the condition that every curve $c(t) = \exp(th)$, for every $h \in H_m$, represents the parabolic arrow $[c]_H = h \in H_m$. This, in turn, is equivalent to the requirement that there exists a second curve $c_2(t)$ that is everywhere tangent to H and such that $c'_2(0) = c'(0)$ and such that $c''_2(0)$ agrees with $c''(0)$ in the directions transversally to H_m (Proposition 17). We are not sure if this makes things any clearer, but it does bring out, to some extent, the geometric meaning of H -adaptedness. The point is that, while the bundle H may not be integrable, one still wants the exponential map to be such that *rays* in H are mapped to curves in M that ‘osculate’ the bundle H as closely as possible, as measured by the *second* derivative in the transversal direction.

3.2. H -adapted exponential maps and connections. Further geometric insight in the distinctive features of H -adapted exponential maps is obtained if we consider exponential maps that arise from connections. This consideration is also useful because it implies the *existence* of H -adapted exponential maps.

Observe that the choice of a distribution $H \subseteq TM$ is equivalent to a reduction of the principal frame bundle fTM to the subbundle fT_HM , whose fiber at $m \in M$ consists of frames (e_1, \dots, e_n) in T_mM for which (e_1, \dots, e_p) is a frame in H_m . (A local section of fT_HM is what we have called an H -frame.) The fiber bundle fT_HM is a principal bundle with structure group,

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \text{End}(\mathbb{R}^p \oplus \mathbb{R}^q) \right\} \subseteq GL_{p+q}(\mathbb{R}).$$

In other words, a Heisenberg structure on M is equivalent to a G -structure on TM , and the natural connections to consider are connections on the principal G -bundle fT_HM . For the associated affine connection ∇ on TM , this simply means that if $X \in \Gamma(H)$, then $\nabla_Y X \in \Gamma(H)$, in other words, ∇ is a connection on TM that can be restricted to a connection on H . One easily verifies that this last condition implies that the exponential map $\exp^\nabla : TM \rightarrow M$ is H -adapted. We abuse the notation \exp^∇ for the induced exponential map on parabolic arrows,

$$\exp^\nabla : T_HM \rightarrow M,$$

even though this also involves choosing an explicit section $j : N \rightarrow TM$. In practice one may choose to work only with H -adapted exponential maps induced by G -connections. This approach has the advantage that it does not require the notion of parabolic arrows. However, it is insufficient to encompass the existing definitions of osculating structures found in the literature. We will see, for example, that the osculating structures of Beals and Greiner, while they do arise from H -adapted exponential maps, do not arise from G -connections.

3.3. H -adapted exponential maps from H -coordinates. A second method for constructing H -adapted exponential maps is by means of a smooth system of H -coordinates. A smooth system of H -coordinates is a choice of H -coordinates

$$E_m : \mathbb{R}^n \rightarrow M,$$

at each point $m \in M$, in such a way that the map $(m, x) \mapsto E_m(x)$ is smooth. Observe that, in general, such a smooth system will only exist *locally*, and the exponential maps defined by means of such a system of coordinates are likewise only locally defined. Such a local definition suffices for the purpose of specifying the Heisenberg calculus.

Let

$$F_m : \mathbb{R}^n \rightarrow T_HM_m$$

denote the Taylor coordinates induced by the H -coordinates E_m . It is immediately clear from the definition of Taylor coordinates (Definition 4) that the composition

$$\exp_m = E_m \circ F_m^{-1} : T_H M_m \rightarrow M$$

is an H -adapted exponential map.

This particular way of obtaining an H -adapted exponential map, while geometrically more clumsy, is most useful in explaining existing definitions of osculating group structures in the literature. The map E_m identifies the coordinate space \mathbb{R}^n with a neighborhood of a point m in the manifold M , while F_m identifies the same coordinate space with the group $T_H M_m$. The typical procedure of analysts is to specify explicit formulas for a group structure on the coordinate space \mathbb{R}^n . But in each case their procedure is clarified if we identify their formulas as specific instances of our Taylor coordinates F_m . We will analyze two exemplary cases.

The osculating structures of Folland and Stein. Osculating structures on contact manifolds first appeared in the work of Folland and Stein (see [6], sections 13 and 14). The construction of Folland and Stein can be summarized as follows (we generalize slightly).

On a $(2k+1)$ -dimensional contact manifold, with $2k$ -dimensional bundle $H \subseteq TM$, choose a (local) H -frame X_1, \dots, X_{2k+1} . It can be shown that this frame can be chosen such that

$$\begin{aligned} [X_i, X_{k+i}] &= X_{2k+1} \bmod H, \text{ for } i = 1, \dots, p, \\ [X_i, X_j] &= 0 \bmod H, \text{ for all other values of } i \leq j. \end{aligned}$$

Then, for $v \in \mathbb{R}^{2k+1}$, let $E_m(v)$ be the endpoint $c(1)$ of the integral curve $c(t)$ of the vector field $\sum v_i X_i$ with $c(0) = m$, in other words,

$$E_m : \mathbb{R}^{2k+1} \rightarrow M : v \mapsto \Phi_{\sum v_i X_i}^1(m),$$

where Φ_Y^t denotes the flow generated by a vector field Y . For Folland and Stein, the ‘osculating Heisenberg structure’ on M is the family of maps E_m , identifying an open subset of the Heisenberg group $H_k = \mathbb{R}^{2k+1}$ (with its standard coordinates) with a neighborhood of $m \in M$.

In the framework we have established here, we see that the commutator relations for the H -frame allow us to identify the basis $X_i(m) \in H_m \oplus N_m$ of the osculating Lie algebra with the standard basis of the Lie algebra of the Heisenberg group. Accordingly, we have a group isomorphism,

$$F_m : H_k = \mathbb{R}^{2k+1} \rightarrow H_m \oplus N_m \rightarrow T_H M_m : v \mapsto \sum v_i X_i(m) \mapsto \exp(\sum v_i X_i(m)).$$

We recognize F_m as the Taylor coordinates on $T_H M_m$ for the H -coordinates E_m at m . We see that the osculating structure on M , as defined by Folland and Stein, can be interpreted as the H -adapted exponential map $\exp_m = E_m \circ F_m^{-1}$.

Alternatively, we can conceptualize the construction of Folland and Stein as specifying an H -adapted exponential map by means of a (locally defined) *flat* G -connection ∇ on TM . The connection ∇ is the one for which the H -frame X_1, \dots, X_n is parallel.

The Heisenberg manifolds of Beals and Greiner. Our second example is the group structure defined by Beals and Greiner on the coordinate space for given H -coordinates $E_m : \mathbb{R}^n \rightarrow U$ at a point m (see [1], section 1.1). Beals and Greiner only consider the case where $H \subset TM$ is an arbitrary distribution of codimension one. Proposition 12 provides an explicit formula for composition in the osculating group $T_H M_m$ in terms of the Taylor coordinates $F_m : \mathbb{R}^n \cong T_H M_m$. This formula, when specialized to the case $k = 1$, agrees exactly with the unexplained string of formulas (1.8), (1.11), (1.15) in [1]. Those formulas can thus be reinterpreted quite simply as the coordinate expression of an H -adapted exponential map $\exp_m = E_m \circ F_m^{-1}$.

If we compare these two examples, we see that Beals and Greiner start their construction with an arbitrary system of H -coordinates E_m , and are therefore required to compensate by a

quadratic correction term in the Taylor coordinates F_m . Folland and Stein, on the other hand, choose a very specific type of coordinates E_m that is better suited to the Heisenberg structure, and as a result obtain Taylor coordinates F_m that are simply the linear coordinates on $H_m \oplus N_m$. Both strategies can be used in the case of more general distributions $H \subseteq TM$.

4. THE PARABOLIC TANGENT GROUPOID

In [4] section II.5, Connes introduces the tangent groupoid as part of a streamlined proof of the Atiyah-Singer index theorem. The convolution algebra of this groupoid encodes the quantization of symbols (as functions on T^*M) to classical pseudodifferential operators. The smoothness of the tangent groupoid proves, in an elegant geometric way, that the principal symbol of a classical pseudodifferential operator is invariantly defined as a distribution on TM (or, equivalently, its Fourier transform on T^*M).

In this section we discuss the definition of a tangent groupoid that plays the same role for the Heisenberg calculus. Our notion of parabolic arrows immediately suggests a topology for this groupoid. But our proof that the groupoid has a well-defined smooth structure relies most crucially on the notion of H -adapted exponential maps. The succesful definition of a tangent groupoid by means of H -adapted maps can be considered as proof that the principal symbol in the Heisenberg calculus is well-defined. More specifically, it proves that we can use an arbitrary H -exponential map to pull back Schwartz kernels from $M \times M$ to $T_H M$ when we develop the calculus.

4.1. The parabolic tangent groupoid and its topology. As a generalization of Connes' tangent groupoid, which relates the total space of the tangent bundle TM to the pair groupoid $M \times M$, we define a similar groupoid in which the bundle of abelian groups TM is replaced by the fiber bundle of osculating groups $T_H M$. We shall refer to this groupoid as the *parabolic tangent groupoid* of a manifold with distribution $H \subseteq TM$, and denote it by $\mathbb{T}_H M$.

As an algebraic groupoid, $\mathbb{T}_H M$ is the disjoint union,

$$\mathbb{T}_H M = \left(\bigcup_{t \in (0,1]} \mathcal{G}_t \right) \cup \left(\bigcup_{m \in M} \mathcal{G}_m \right),$$

of a parametrized family of pair groupoids with the collection of osculating groups,

$$\begin{aligned} \mathcal{G}_t &= M \times M, \quad t \in (0, 1], \\ \mathcal{G}_m &= T_H M_m, \quad m \in M. \end{aligned}$$

Clearly, the union $\bigcup \mathcal{G}_t = M \times M \times (0, 1]$ by itself is a smooth groupoid, and the same is true, as we have seen, for the bundle of osculating groups $\bigcup \mathcal{G}_m = T_H M$. We write $\mathcal{G}_0 = T_H M$, and $\mathcal{G}_{(0,1]} = M \times M \times (0, 1]$. Each groupoid $\mathcal{G}_t, t \in [0, 1]$ has object space M , and the object space for the total groupoid $\mathcal{G} = \mathbb{T}_H M$ is the manifold,

$$\mathcal{G}^{(0)} = M \times [0, 1].$$

We will endow $\mathbb{T}_H M$ with the structure of a manifold with boundary, by glueing \mathcal{G}_0 as the $t = 0$ boundary to $\mathcal{G}_{(0,1]}$. The topology on $\mathbb{T}_H M$ is such that $\mathcal{G}_{(0,1]}$ is an open subset of $\mathbb{T}_H M$. The topology on $\mathbb{T}_H M$ can be defined very nicely by means of our parabolic arrows. The construction in [4, II.5] generalizes immediately if we replace tangent vectors by parabolic arrows.

Definition 20. A curve $(a(t), b(t), t)$ in $\mathcal{G}_{(0,1]} = M \times M \times (0, 1]$ converges, as $t \rightarrow 0$, to a parabolic arrow $(m, v) \in T_H M$ if,

$$\begin{aligned} M \ni m &= \lim_{t \rightarrow 0} a(t) = \lim_{t \rightarrow 0} b(t), \\ T_H M_m \ni v &= [a]_H * [b]_H^{-1}, \end{aligned}$$

where we assume that $a'(0) \in H$ and $b'(0) \in H$.

Recall that $[a]_H$ and $[b]_H$ denote the parabolic arrows defined by the curves a, b , while the expression $[a]_H * [b]_H^{-1}$ denotes the product of $[a]_H$ with the inverse of $[b]_H$ in the osculating group $T_H M_m$. If $H = TM$, the osculating groups are abelian, and the definition simplifies to

$$T_m M \ni v = a'(0) - b'(0) = \lim_{t \rightarrow 0} \frac{a(t) - b(t)}{t}.$$

This is precisely the topology of the tangent groupoid as defined by Connes in [4].

It is easy to see that the groupoid operations for $\mathbb{T}_H M$ are continuous. For example, in $\mathcal{G}_{(0,1]}$ we have,

$$(a(t), b(t), t) \cdot (b(t), c(t), t) = (a(t), c(t), t),$$

while in \mathcal{G}_0 ,

$$([a]_H * [b]_H^{-1}) * ([b]_H * [c]_H^{-1}) = [a]_H * [c]_H^{-1},$$

assuming that $a(0) = b(0) = c(0)$ and $a'(0), b'(0), c'(0) \in H_m$. However, we will not rigorously develop this point of view. Instead, we glue \mathcal{G}_0 to $\mathcal{G}_{(0,1]}$ in an alternative way, more convenient for practical use, by defining a smooth structure on $\mathbb{T}_H M$.

4.2. Charts on the parabolic tangent groupoid. We now define charts on the groupoid $\mathbb{T}_H M$ by means of exponential maps, completely analogous to the construction in [4, II.5].

Suppose we have an H -adapted exponential map,

$$\exp : T_H M \rightarrow M.$$

We define a map,

$$\psi : T_H M \times [0, 1) \rightarrow \mathbb{T}_H M,$$

by,

$$\begin{aligned} \psi(m, v, t) &= (\exp_m(\delta_t v), m, t), \text{ for } t > 0, \\ \psi(m, v, 0) &= (m, v) \in T_H M_m. \end{aligned}$$

Here δ_t denotes the Heisenberg dilation in the osculating group $T_H M_m$. The smooth structure on $T_H M \times [0, 1)$ induces a smooth structure in an open neighborhood of $\mathcal{G}_0 = T_H M$ in $\mathbb{T}_H M$.

At first sight it may seem that the main modification to the construction of the classical tangent groupoid is the introduction of the parabolic dilations $\delta_t v$ to replace the simple ‘blow-up’ tv of Connes. This simple idea is indeed the obvious thing to do if one wants to define a tangent groupoid for the Heisenberg calculus. But the real crux of the definition of the parabolic tangent groupoid is the requirement that the exponential map must be H -adapted. If one uses arbitrary exponential maps to define the charts on $\mathbb{T}_H M$, in combination with the parabolic dilations δ_t , then the resulting charts are *not* smoothly compatible (i.e., transition functions would not be smooth). It is precisely for this reason that arbitrary exponential maps cannot be used to define Heisenberg pseudodifferential operators.

To better understand the glueing of the bundle $T_H M$ to $M \times M \times (0, 1]$ it may be helpful to recall the construction of an H -adapted exponential map by means of a system of H -coordinates (in an open set $U \subseteq M$),

$$E_m : \mathbb{R}^n \rightarrow U.$$

Recall that H -coordinates are such that, for each $m \in U$, the coordinate vectors $dE_m(\partial/\partial x_i)$, for $i = 1, \dots, p$, are vectors in H_m . As before, let

$$F_m : \mathbb{R}^{p+q} \rightarrow T_H M_m$$

denote the Taylor coordinates on the osculating group $T_H M_m$, induced by the H -coordinates E_m (Definition 4). As was discussed, the composition $\exp_m(v) = E_m \circ F_m^{-1}$ is a (local) H -adapted

exponential map. From this perspective, equivalent to the chart ψ that was defined above we could work with the chart,

$$\psi' : U \times \mathbb{R}^p \times \mathbb{R}^q \times [0, 1) \rightarrow \mathbb{T}_H M,$$

defined by,

$$\begin{aligned}\psi'(m, h, n, t) &= (E_m(th, t^2 n), m, t), \text{ for } t > 0, \\ \psi'(m, h, n, 0) &= F_m(m, h, n) \in T_H M_m.\end{aligned}$$

This description brings out very clearly what is going on. The expression $(E_m(th, t^2 n), m, t)$ for $t > 0$ corresponds to a ‘blow up’ of the diagonal in $M \times M$ by a factor t^{-1} in the direction of H , and by a factor t^{-2} in the direction transversal to H . This is precisely what one would expect.

But the success of the construction crucially depends on the choice of coordinates at $t = 0$, involving the Taylor coordinates F_m . And this is the subtle ingredient in the construction of the groupoid. Recall that, if we make the *canonical* identification of $T_H M$ with the bundle $H \oplus N$ (by means of the Lie exponential map in the fibers), then the Taylor coordinates F_m are explicitly given by,

$$\log F_m(m, h, n) = (h, n - \tfrac{1}{2}b_m(h, h)) \in H_m \oplus N_m,$$

where $b_m(h, h)$ is a quadratic form that depends on the coordinates E_m (see Propositions 12 and 24). The necessity and nature of this quadratic correction term $b_m(h, h)$ would be very hard to guess if one had to construct the parabolic tangent groupoid from scratch.

If the coordinates E_m are chosen in such a way that the corresponding bilinear form b_m is skew-symmetric (for example, as in the construction of Folland and Stein), then this quadratic correction term vanishes, and we can work simply with the natural coordinates on $H_m \oplus N_m$ at $t = 0$. Correspondingly, an alternative solution to the construction of the parabolic tangent groupoid would be to work with ‘preferred’ coordinate systems E_m , i.e., H -coordinates for which b_m is skew-symmetric.

4.3. Proof that the smooth structure is well-defined. We now show that, with the above choices, the manifold structure on $\mathbb{T}_H M$ is well defined. The proofs of Propositions 22 and 23 show the relevance of the corrected groupoid coordinates at $t = 0$, if arbitrary H -coordinates E_m are allowed. The basic ingredient of the proof is the following technical lemma.

Lemma 21. *Let $\phi : T_H M \rightarrow T_H M$ be a diffeomorphism that preserves the fibers; fixes the zero section $M \subset T_H M$; and at the point m has derivative $D\phi_m = \text{id}$, and a second derivative that satisfies $D^2\phi_m(h, h) \in H_m$, for $h \in H_m$. Then the map,*

$$\tilde{\phi} : T_H M \times \mathbb{R} \rightarrow T_H M \times \mathbb{R},$$

defined by,

$$\begin{aligned}\tilde{\phi}(m, v, t) &= (\delta_t^{-1}\phi(m, \delta_t v), t), \\ \tilde{\phi}(m, v, 0) &= (m, v, 0),\end{aligned}$$

is a diffeomorphism.

Proof. Clearly, $\tilde{\phi}$ is smooth on the open subset where $t \neq 0$. We must prove that $\tilde{\phi}$ is smooth in a neighborhood of the $t = 0$ fiber.

For convenience of notation, we identify $T_H M$ with $H \oplus N$ via the logarithm. The proof is based on a simple Taylor expansion near $t = 0$. For a choice of coordinates on $H \oplus N$ we have,

$$\phi(m, v) = \phi(m, 0) + D\phi_m(v) + \frac{1}{2}D^2\phi_m(v, v) + R(m, v).$$

The remainder term $R = R(m, v)$ satisfies a bound $|R| < C|v|^3$, for $|v| < 1$. Now write $v = h + n$ with $h \in H_m, n \in N_m$. Then,

$$\begin{aligned}\phi(m, th + t^2n) &= \phi(m, 0) + tD\phi_m(h) + t^2D\phi_m(n) \\ &\quad + \frac{1}{2}t^2D^2\phi_m(h, h) + t^3D^2\phi_m(h, n) + \frac{1}{2}t^4D^2\phi_m(n, n) + R(m, th + t^2n) \\ &= \phi(m, 0) + tD\phi_m(h) + t^2D\phi_m(n) + \frac{1}{2}t^2D^2\phi_m(h, h) + t^3R'.\end{aligned}$$

The error term $R' = r'(m, h, n, t)$, satisfies a bound $|R'| \leq C$ for $|h| < \epsilon|t|^{-1}$, $|n| < \epsilon|t|^{-2}$. Observe that these inequalities hold in an open neighborhood of the $t = 0$ fiber in $T_H M \times \mathbb{R}$.

The assumptions on ϕ allow the simplification,

$$\phi(m, \delta_t v) = (m, th + t^2n + \frac{1}{2}t^2D^2\phi_m(h, h) + t^3R'),$$

where $D^2\phi_m(h, h) \in H_m$. We find,

$$\delta_t^{-1}\phi(m, \delta_t v) = (m, v + tR''),$$

where, again, the coefficient of the remainder R'' is uniformly bounded in a neighborhood of the $t = 0$ fiber. This implies continuity of ϕ .

By the same reasoning, expanding ϕ in a higher order Taylor series, one obtains,

$$\tilde{\phi}(m, v, t) = (m, v + \sum_{k=1}^r a_k t^k + R_r t^r, t),$$

where the coefficients $a_k = a_k(m, v)$ are smooth functions, independent of t , arising from the derivatives of ϕ , while the coefficient R_r of the remainder is uniformly bounded in a neighborhood of $t = 0$. This implies smoothness of $\tilde{\phi}$. □

Proposition 22. *For different choices of H -adapted exponential maps $\exp: T_H M \rightarrow M$ the charts $\psi: T_H M \times [0, 1] \rightarrow \mathbb{T}_H M$, defined above, have smooth transition functions. In other words, $\mathbb{T}_H M$ has a well-defined structure of smooth manifold, independent of the choice of H -adapted exponential map.*

Proof. Let ψ and ψ' be the two maps,

$$\psi, \psi' : T_H M \times [0, 1] \rightarrow \mathbb{T}_H M,$$

constructed in the manner explained above, for two different exponential maps $E, E' : T_H M \rightarrow M$. We must prove that the transition function $\tilde{\phi} = \psi^{-1} \circ \psi'$ is smooth. We have,

$$\begin{aligned}\tilde{\phi}(m, v, t) &= (\delta_t^{-1}E_m^{-1}(E'_m(\delta_t v)), m, t), \text{ for } t \neq 0, \\ \tilde{\phi}(m, v, 0) &= (m, v, 0).\end{aligned}$$

Definition 19 of H -adapted exponential maps immediately implies that the composition $\phi_m = E_m^{-1} \circ E'_m$ satisfies the assumptions of Lemma 21. Hence, $\tilde{\phi}$ is smooth. □

4.4. Compatibility of smooth structure and topology. The next proposition shows that the manifold structure on $\mathbb{T}_H M$ is compatible with the topology according to Definition 20.

Proposition 23. *Suppose $a(t), b(t)$ are smooth curves in M with $a(0) = b(0) = m$, such that $a'(0)$ and $b'(0)$ are in H_m . Then in $\mathbb{T}_H M$, endowed with the manifold structure defined above,*

$$\lim_{t \rightarrow 0} (a(t), b(t), t) = [a]_H * [b]_H^{-1} \in T_H M_m.$$

First proof. If we assume that the curve $(a(t), b(t), t)$ in $\mathcal{G}_{(0,1]}$ extends to a smooth curve in $\mathbb{T}_H M$, then there is a nice proof that makes use of parabolic flows. Let $v_0 \in T_H M_m$ be the point in \mathcal{G} to which the curve in $\mathcal{G}_{(0,1]}$ converges, and let v_t be the parabolic arrow defined by, $\psi(b(t), v_t, t) = (a(t), b(t), t)$, i.e.,

$$a(t) = \exp_{b(t)}(\delta_t v_t).$$

By definition of the manifold structure on $\mathbb{T}_H M$, we have $(b(t), v_t) \rightarrow (m, v_0)$. We see that the section $v_t, t \in [0, 1]$ is smooth along $b(t)$, and can be extended to a section V in a neighborhood of $m = b(0)$. Now define a flow,

$$\Phi_v^t(m') = \exp_{m'}(\delta_t V(m')).$$

By definition of Heisenberg exponential maps, the curve,

$$t \mapsto \exp_{m'}(\delta_t V(m'))$$

has parabolic arrow $V(m')$. In other words, Φ_v^t is a parabolic flow, and in particular,

$$[\Phi_v^t(m)]_H = V(m) = v_0 \in T_H M_m.$$

Clearly $a(t) = \Phi_v^t(b(t))$. Extend $b(t)$ to a parabolic flow Φ_b^t , such that $b(t) = \Phi_b^t(m)$. Then we see that,

$$[a]_H = [\Phi_v^t \circ \Phi_b^t(m)]_H = [\Phi_v^t(m)]_H * [\Phi_b^t(m)]_H = v_0 * [b]_H,$$

which means that $v_0 = [a]_H * [b]_H^{-1}$.

□

Second proof. To prove the proposition without the extra assumption of convergence, and to illustrate a different technique, we give a second proof.

We use the map ψ' defined above to describe the manifold structure on $\mathbb{T}_H M$. We need a system of H -coordinates E_m , and the corresponding Taylor coordinates F_m . Let us identify an open set $U \subseteq M$ with \mathbb{R}^n (via a coordinate map that we suppress in the notation). Given an H -frame X_i on U , we have a system of coordinates,

$$E_m : \mathbb{R}^n \rightarrow U : v \mapsto m + \sum v_i X_i(m) = m + Xv.$$

Here $X = (X_i^j)$ denotes the $n \times n$ matrix whose columns are the vector-values functions $X_i : U \rightarrow \mathbb{R}^n$.

Expand a and b in the coordinates on $U \cong \mathbb{R}^n$ as,

$$\begin{aligned} a(t) &= th + t^2 k + \mathcal{O}(t^3), \\ b(t) &= th' + t^2 k' + \mathcal{O}(t^3), \end{aligned}$$

assuming that $a(0) = b(0) = 0$. We have Taylor coordinates,

$$F_m(h, n) = [a]_H, \quad F_m(h', n') = [b]_H,$$

where $n = k^N$, $n' = k'^N$ are the normal components of k, k' . With the notation of Proposition 12, we compute,

$$\begin{aligned} F_m^{-1}([a]_H * [b]_H^{-1}) &= (h, n) * (h', n')^{-1} \\ &= (h, n) * (-h', -n' + b(h', h')) \\ &= (h - h', n - n' - b(h, h') + b(h', h')). \end{aligned}$$

Now let $(a(t), b(t), t) = \psi'(b(t), x(t), y(t), t)$, i.e.,

$$a(t) = E_{b(t)}(tx(t), t^2 y(t)),$$

where the coordinates $(x(t), y(t)) \in \mathbb{R}^{p+q}$ depend on t . We must show that,

$$\lim_{t \rightarrow 0} (x(t), y(t)) = (-h + h', -n + n' - b(h, h') + b(h, h)).$$

We approximate the coordinates $(x(t), y(t)) \in \mathbb{R}^{p+q}$ by a Taylor expansion of $E_{b(t)}^{-1}(a(t))$, using the explicit form of E_m , as follows,

$$\begin{aligned} (tx(t), t^2y(t)) &= X_{b(t)}^{-1}(a(t) - b(t)) \\ &= a(t) - b(t) + t D(X^{-1})_0(h', a(t) - b(t)) + \mathcal{O}(t^3) \\ &= t(h - h') + t^2(k - k') + t^2 D(X^{-1})_0(h', h - h') + \mathcal{O}(t^3), \end{aligned}$$

Let us explain the calculation. In the first step we expanded $X^{-1}(b(t))$. Because $a(t) - b(t) = \mathcal{O}(t)$, it sufficed to consider only the first derivative,

$$\frac{\partial}{\partial t} X^{-1}(b(t))|_{t=0} = D(X^{-1})_0 \cdot h'.$$

In the second step we expanded $a(t) - b(t)$, again ignoring terms of order $\mathcal{O}(t^3)$.

Reversing the dilation, we find,

$$(x(t), y(t)) = (h - h', n - n' + D(X^{-1})_0^N(h', h - h')) + \mathcal{O}(t).$$

Because $X_0 = 1$, we have $D(X^{-1})_0 = -DX_0$, while the normal component $DX_0^N(h', h - h')$ is equal to $b(h', h - h')$, by definition of the bilinear form b . This gives the desired result. \square

APPENDIX: TWO-STEP NILPOTENT GROUPS.

We collect here some simple facts about two-step nilpotent groups that play a role in this paper. These facts are elementary, but we are not aware of a reference that contains the formulas we need.

Recall that a Lie algebra \mathfrak{g} is called *two-step nilpotent* if $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$. The Campbell–Baker–Hausdorff formula for such Lie algebras has very few non-zero terms:

$$\exp(x) \cdot \exp(y) = \exp\left(x + y + \frac{1}{2}[x, y]\right),$$

for $x, y \in \mathfrak{g}$. Replacing the bracket $[x, y]$ with an arbitrary (not necessarily skew-symmetric) bilinear map $B : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^q$, we can generalize and define a Lie group $G_B = \mathbb{R}^p \times \mathbb{R}^q$ with group operation

$$(h_1, n_1) * (h_2, n_2) = (h_1 + h_2, n_1 + n_2 + B(h_1, h_2)).$$

It is trivial to verify the group axioms (using the bilinearity of B). By Proposition 12, the group structure of parabolic arrows $T_H M_m$ expressed in Taylor coordinates is of this type. Our main goal in this section is to prove the following proposition.

Proposition 24. *Let G_B be the Lie group defined above. With the natural coordinates on $G_B = \mathbb{R}^{p+q}$ and $\text{Lie } G_B = T_0 \mathbb{R}^{p+q}$, the exponential map $\exp : \text{Lie } G_B \rightarrow G_B$ is expressed as*

$$\exp(h, n) = (h, n + \frac{1}{2}B(h, h)).$$

The proof consists of a string of lemmas.

Lemma 25. *The Lie algebra structure on $\text{Lie } G_B = \mathbb{R}^{p+q}$ is given by the bracket*

$$[(h_1, n_1), (h_2, n_2)] = (0, B(h_1, h_2) - B(h_2, h_1)).$$

In particular, the Lie algebra structure only depends on the skew-symmetric part $(B - B^T)/2$ of the bilinear map B .

Proof. The neutral element in G_B is $(0, 0)$, and inverses are given by

$$(h, n)^{-1} = (-h, -n + B(h, h)).$$

Commutators in G_B are calculated as follows:

$$\begin{aligned} & (h_1, n_1) * (h_2, n_2) * (h_1, n_1)^{-1} * (h_2, n_2)^{-1} \\ &= (h_1, n_1) * (h_2, n_2) * (-h_1, -n_1 + B(h_1, h_1)) * (-h_2, -n_2 + B(h_2, h_2)) \\ &= (h_1 + h_2, n_1 + n_2 + B(h_1, h_2)) * \\ & \quad (h_1 - h_2, -n_1 - n_2 + B(h_1, h_1) + B(h_2, h_2) + B(-h_1, -h_2)) \\ &= (0, B(h_1, h_1) + B(h_2, h_2) + 2B(h_1, h_2) + B(h_1 + h_2, -h_1 - h_2)) \\ &= (0, B(h_1, h_2) - B(h_2, h_1)). \end{aligned}$$

Replace (h_i, n_i) with (th_i, tn_i) and take the limit as $t \rightarrow 0$. □

We see that the groups G_B are indeed two-step nilpotent, or even abelian in the trivial case where B is symmetric.

Lemma 26. *If $B: \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ is a skew-symmetric bilinear map, then the exponential map $\exp: \text{Lie}(G_B) \rightarrow G_B$ is the usual identification of $T_0\mathbb{R}^n$ with \mathbb{R}^n .*

Proof. For any $(h, n) \in \mathbb{R}^{p+q}$ we have $(th, tn) * (sh, sn) = ((t+s)h, (t+s)n)$. In other words, the map

$$\phi: \mathbb{R} \rightarrow G_b: t \mapsto (th, tn)$$

is a group homomorphism. The tangent vector to this one-parameter subgroup at $t = 0$ is $\phi'(0) = (h, n) \in \text{Lie}(G_B)$, and by definition $\exp(\phi'(0)) = \phi(1) = (h, n) \in G_B$. □

Lemma 27. *If $B, C: \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ are two bilinear maps that have the same skew-symmetric part, then the quadratic map*

$$\phi: G_C \xrightarrow{\cong} G_B: (h, n) \mapsto (h, n + \frac{1}{2}B(h, h) - \frac{1}{2}C(h, h)),$$

is a group isomorphism.

Proof. With $S = B - C$:

$$\begin{aligned} & \phi(h_1, n_1) * \phi(h_2, n_2) \\ &= (h_1, n_1 + \frac{1}{2}S(h_1, h_1)) * (h_2, n_2 + \frac{1}{2}S(h_2, h_2)) \\ &= (h_1 + h_2, n_1 + \frac{1}{2}S(h_1, h_1) + n_2 + \frac{1}{2}S(h_2, h_2) + C(h_1, h_2)) \\ &= (h_1 + h_2, n_1 + n_2 + \frac{1}{2}S(h_1, h_1) + \frac{1}{2}S(h_2, h_2) + S(h_1, h_2) + B(h_1, h_2)) \\ &= (h_1 + h_2, n_1 + n_2 + \frac{1}{2}S(h_1 + h_2, h_1 + h_2) + B(h_1, h_2)) \\ &= \phi(h_1 + h_2, n_1 + n_2 + B(h_1, h_2)) = \phi((h_1, n_1) * (h_2, n_2)). \end{aligned}$$

□

Proof of Proposition 24. Let $C = \frac{1}{2}(B - B^T)$ be the skew-symmetric part of B . The exponential map for G_B is the composite of the following three maps:

$$\text{Lie}(G_B) \xrightarrow{\cong} \text{Lie}(G_C) \xrightarrow{\exp} G_C \xrightarrow{\phi} G_B.$$

The first two of these maps are just the identity map $\mathbb{R}^{p+q} \rightarrow \mathbb{R}^{p+q}$ (by Lemmas 25 and 26, respectively). Lemma 27 gives the explicit isomorphism $\phi: G_C \cong G_B$, with $C(h, h) = 0$. □

REFERENCES

- [1] R. Beals and P. Greiner, *Calculus on Heisenberg manifolds*, Annals of Mathematics Studies, vol. 119, Princeton University Press, Princeton, NJ, 1988.
- [2] A. Bellaïche and J.-J. Risler (eds.), *Sub-Riemannian geometry*, Progress in Mathematics, vol. 144, Birkhäuser Verlag, Basel, 1996. MR1421821 (97f:53002)
- [3] E. Cartan, *Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. École Norm. Sup. (3) **27** (1910), 109–192. MR1509120
- [4] A. Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994. MR1303779 (95j:46063)
- [5] T. E. Cummins, *A pseudodifferential calculus associated to 3-step nilpotent groups*, Comm. Partial Differential Equations **14** (1989), no. 1, 129–171. MR973272 (90c:58176)
- [6] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.
- [7] T. Morimoto, *Lie algebras, geometric structures and differential equations on filtered manifolds*, Lie groups, geometric structures and differential equations—one hundred years after Sophus Lie (Kyoto/Nara, 1999), 2002, pp. 205–252. MR1980903 (2004e:58031)
- [8] R. Ponge, *The tangent groupoid of a Heisenberg manifold*, Pacific J. Math. **227** (2006), no. 1, 151–175. MR2247877 (2007d:58025)
- [9] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), no. 3-4, 247–320.
- [10] M. E. Taylor, *Noncommutative microlocal analysis. I*, Mem. Amer. Math. Soc. **52** (1984), no. 313, iv+182.
- [11] E. van Erp, *The Atiyah-Singer formula for subelliptic operators on a contact manifold, Part I*, Ann. of Math. **171** (2010), 1647–1681.

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